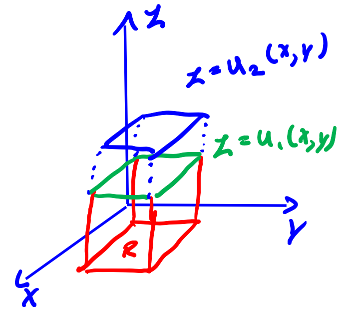
Sec 15.6 Triple Integrals



DEF: Let $f(x, y, z)$ be a continuous function on the solid region

$$E = \{(x, y, z) : (x, y) \in \mathcal{R},\ u_1(x, y) \leq z \leq u_2(x, y)\}$$

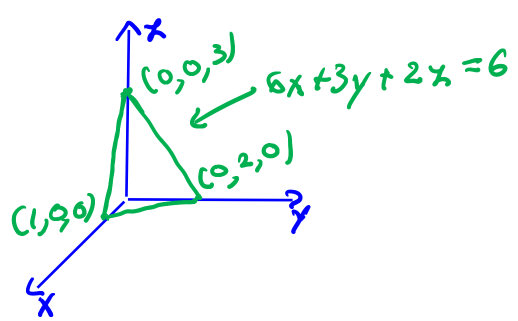
We define the triple integral of $f(x, y, z)$ over E by:

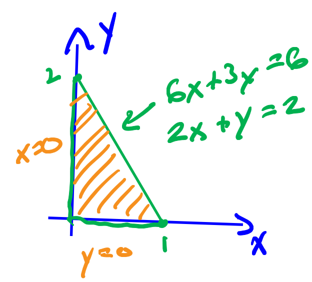
$$\iiint_E f(x,y,z)\, dV := \iint_{\mathcal{R}} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right] dA$$

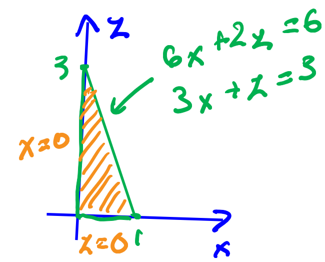
Important: Due to Fubini's theorem, this triple integral can be written in 6 distinct forms.

Ex1. Let E be the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$ and $6x + 3y + 2z = 6$.

(a) Draw the solid region E.







(b) Rewrite the integral $\iiint_E f(x, y, z)\, dz\, dy\, dx$ in the order $dy\, dx\, dz$.

$dz\,dy\,dx$

$z \begin{cases} \text{Bottom surface}: z = 0 \\ \text{Top surface}: z = \dfrac{6-6x-3y}{2} \end{cases}$

$$\int_{x=0}^{1}\int_{y=0}^{2-2x}\int_{0}^{\frac{6-6x-3y}{2}} f(x,y,z)\, dz\, dy\, dx$$

see projection onto the xy-plane

$dy\,dx\,dz$

$y \begin{cases} \text{Bottom surface}: y = 0 \\ \text{Top surface}: y = \dfrac{6-6x-2z}{3} \end{cases}$

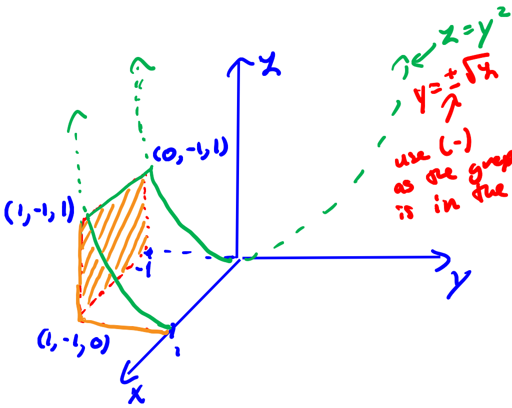
$$\int_{z=0}^{3}\int_{x=0}^{\frac{3-z}{3}}\int_{y=0}^{\frac{6-6x-2z}{3}} f(x,y,z)\, dy\, dx\, dz$$

see projection onto the xz-plane

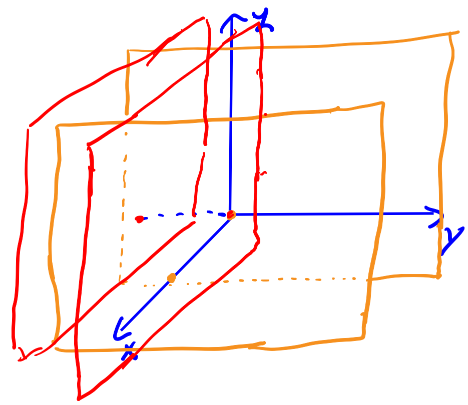
Ex2. Rewrite the following integral using the order of integration $dy\,dz\,dx$

$$\int_0^1 \int_{-1}^0 \int_0^{y^2} z^2 \, dz\, dy\, dx.$$

What is the value of the integral?

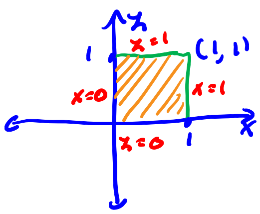


$$\begin{cases} 0 \le x \le 1 \\ -1 \le y \le 0 \\ 0 \le z \le y^2 \end{cases}$$



Projection onto the xz-plane

Bounds for y $\begin{cases} \text{Bottom surface: } y = -1 \text{ (plane)} \\ \text{Top surface: } y = -\sqrt{z} \text{ (parabolic cylinder)} \end{cases}$



then, $I = \displaystyle\int_{x=0}^{1} \int_{z=0}^{1} \int_{y=-1}^{-\sqrt{z}} z^2 \, dy\, \underbrace{dz\, dx}_{\text{see the projection onto the xz-plane}}$

Inner: $\displaystyle\int_{-1}^{-\sqrt{z}} z^2 \, dy = z^2 y \Big|_{y=-1}^{y=\sqrt{z}} = -z^{5/2} - (-z^2) = -z^{5/2} + z^2$

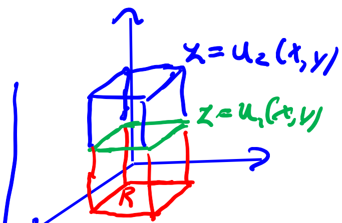
Middle: $\displaystyle\int_0^1 -z^{5/2} + z^2 \, dz = \left(\frac{-z^{7/2}}{7/2} + \frac{z^3}{3} \right) \Big|_{z=0}^{z=1}$

$$= \left(\frac{-2}{7} + \frac{1}{3} \right) - (0+0) = \frac{1}{21}$$

Outer: $\displaystyle\int_0^1 \frac{1}{21}\, dx = \frac{x}{21}\Big|_0^1 = \frac{1}{21} - 0 = \frac{1}{21}$

so, $I = \frac{1}{21}$

$\int_a^b 1\,dt = b-a$ $\qquad$ $\iint_R 1\,dA = $ Area of R $\qquad$ $\iiint_E 1\,dV = $ Volume of E



DEF: The volume of a solid region E is defined by $V(E) := \iiint_E 1\,dV$.

$$\iint_R u_2(x,y)\,dA - \iint_R u_1(x,y)\,dA = \iint_R u_2(x,y)-u_1(x,y)\,dA = \iint_R \left(\int_{u_1(x,y)}^{u_2(x,y)} 1\,dz\right) dA$$

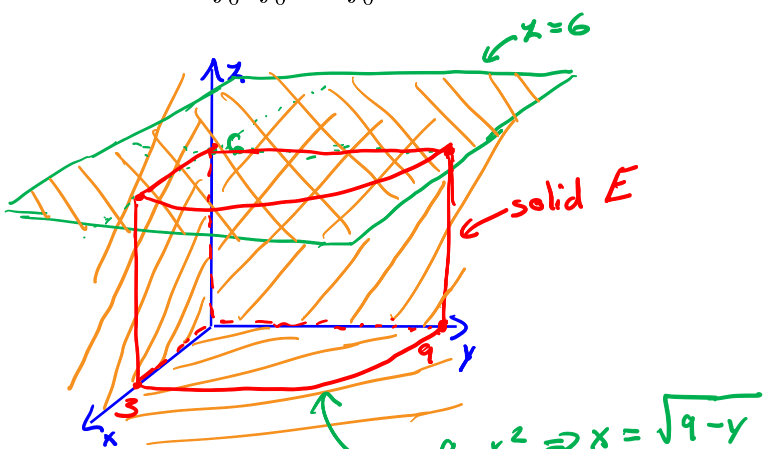
Ex3. A solid E in the first octant is bounded by the planes $x=0$, $y=0$, $z=0$, $z=6$ and the cylinder $y = 9-x^2$. Which of the following integrals represents the volume of the solid E?

~~(a)~~ $\displaystyle\int_0^6 \int_0^{9-x^2} \int_0^{\sqrt{9-y}} dx\,dy\,dz$ ← Impossible

(c) $\displaystyle\int_0^6 \int_0^9 \int_0^{\sqrt{9-y}} dx\,dy\,dz$

~~(b)~~ $\displaystyle\int_0^3 \int_0^{9-x^2} \int_0^{9-x^2-y} dz\,dy\,dx$ ← should be "6" for z

~~(d)~~ $\displaystyle\int_0^3 \int_0^{\sqrt{9-y}} \int_0^6 dz\,dy\,dx$ ← Impossible

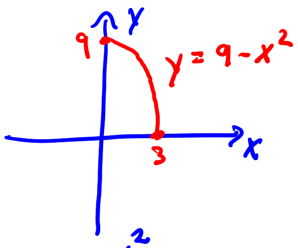


$y = 9-x^2 \Rightarrow x = \sqrt{9-y}$

$dz\,dy\,dx$

$z \begin{cases} \text{bottom surface} & z = 0 \\ \text{top surface} & z = 6 \end{cases}$

projection onto the xy-plane

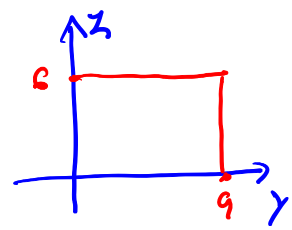


$$\text{Volume} = \int_0^3 \int_0^{9-x^2} \int_0^6 1\,dz\,dy\,dx$$

$dx\,dy\,dz$

$x \begin{cases} \text{bottom surface} & x = 0 \\ \text{top surface} & x = \sqrt{9-y} \end{cases}$

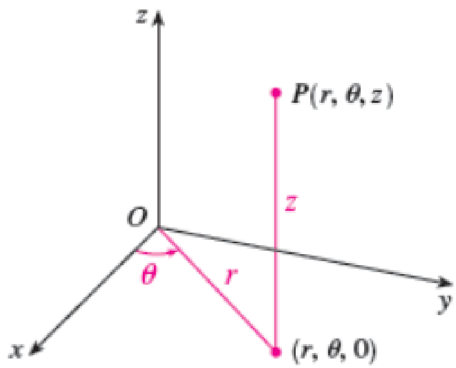
Projection onto the yz-plane

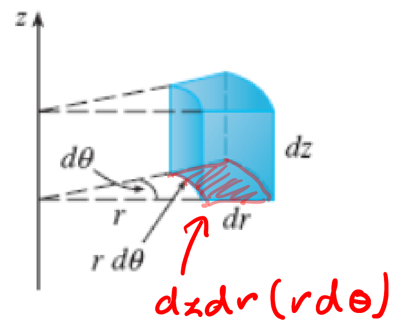


$$\text{Volume} = \int_0^6 \int_0^9 \int_0^{\sqrt{9-y}} 1\,dx\,dy\,dz$$

always need numbers for the outer integral, no variables

Sec 15.7 Triple Integrals in Cylindrical Coordinates





Equations: $x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$

Integration formula:

remember "r"

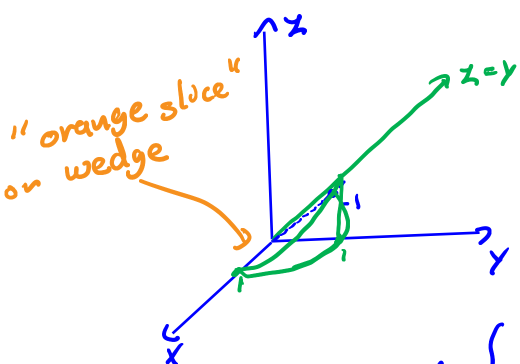
$$\iiint_E f(x,y,z)\;dV = \iiint_{E_{r,\theta,z}} f(r\cos\theta, r\sin\theta, z)\;dz\;r dr\;d\theta$$

projection onto the xy-plane

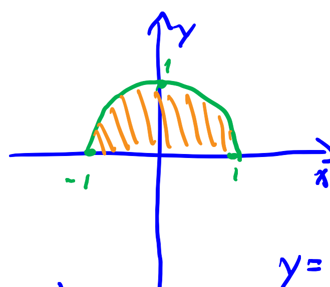
Ex1. Convert the integral

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{y}(x^2+y^2)\;dz\;dy\;dx$$

to an equivalent integral in cylindrical coordinates and evaluate the result.



$$\begin{cases} -1 \le x \le 1 \\ 0 \le y \le \sqrt{1-x^2} \\ 0 \le z \le y \end{cases}$$



$z \begin{cases} \text{bottom surface: } z = 0 \text{ (xy-plane)} \\ \text{top surface: } z = y \Rightarrow \boxed{z = r\sin\theta} \end{cases}$

$y = \sqrt{1-x^2}$

$y^2 = 1 - x^2$

$x^2 + y^2 = 1$

$r^2 = 1$

$r = 1$

then,

remember "r"

$$I = \int_0^{\pi}\int_0^{1}\int_0^{r\sin\theta} r^2\,(r)\;dz\;dr\;d\theta$$

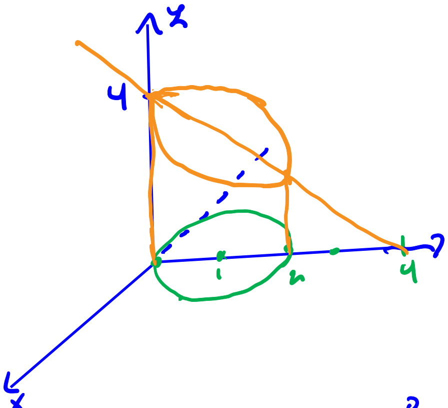
$\Rightarrow$ evaluate $\Rightarrow \boxed{\dfrac{2}{5}}$

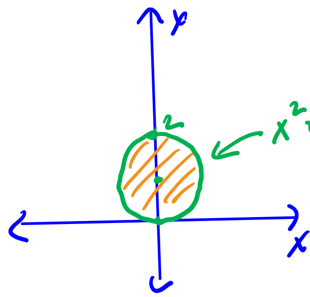
Ex2. Set up the integral for evaluating

$$\iiint_E z \, dV$$

in cylindrical coordinates, where E is the circular solid cylinder whose base is the circle $\underbrace{x^2+y^2}=2y$ in the xy-plane and whose top lies in the plane $z = 4 - y$.

center: complete the square $=(0,1)$





$x^2+y^2=2y$

$\Rightarrow r^2 = 2r\sin\theta \Rightarrow r = 2\sin\theta$

Bounds for z $\begin{cases} \text{Bottom} : & z = 0 \\ \text{Top} : & z = 4 - y \Rightarrow \boxed{z = 4 - r\sin\theta} \end{cases}$

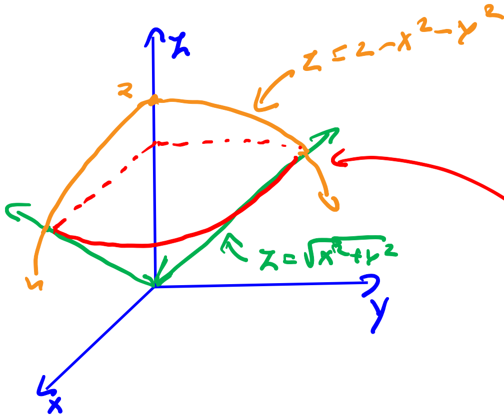
then,

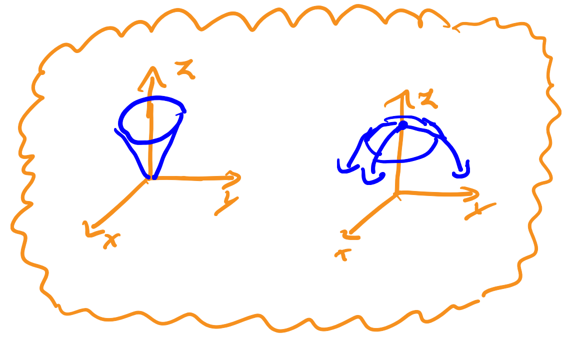
$$I = \int_0^{\pi} \int_0^{2\sin\theta} \int_0^{4-r\sin\theta} z \cdot r \; \underline{dz\, dr\, d\theta}$$

$\hookrightarrow$ check projection into xy-plane

$z^2 = x^2 + y^2$, $z \geq 0$

Ex3. Let E be the solid region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. Set up a triple integral in cylindrical coordinates that gives the volume of E.





Intersection of $z = 2 - x^2 - y^2$ and $z = \sqrt{x^2 + y^2}$

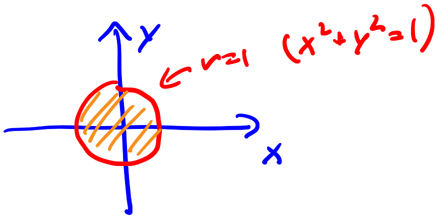
$z = 2 - r^2$ $\qquad$ $z = r$

$\Rightarrow 2 - r^2 = r$

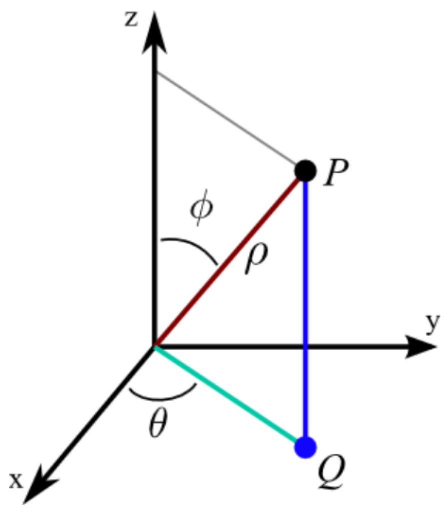
$\Rightarrow r^2 + r - 2 = 0$

$\Rightarrow (r+2)(r-1) = 0$

$\boxed{\cancel{r = -2}}$ or $\boxed{r = 1}$



Sec 15.8 SPHERICAL COORDINATES



Let P be a 3-dimensional point with coordinates (x, y, z). The quantity ρ represents the distance from the point P to the origin. In cartesian coordinates:

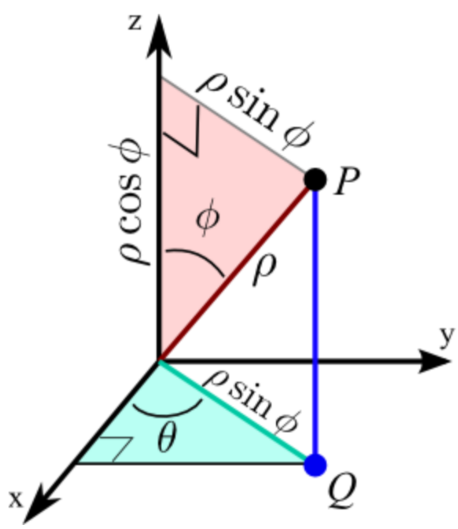
$$\rho = (x^2 + y^2 + z^2)^{1/2}$$

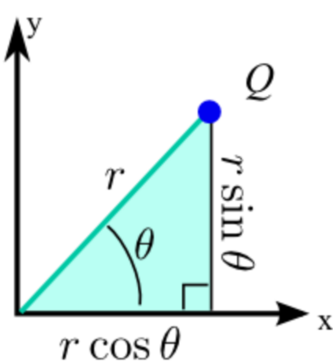
The point Q represents the shadow of the point P on the xy-plane. So Q is the 2-dimensional point (x, y) and the blue segment $\overline{PQ}$ has length equal to z. Thus

$$z = \rho \cos \phi$$

where ϕ is the angle from the positive z-axis to the red segment $\overline{OP}$.

This leads us to





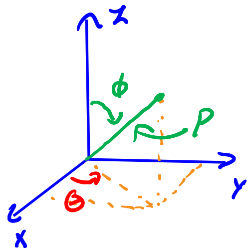
The angle θ is the angle from the positive x-axis to the green segment $\overline{OQ}$. So the pair (r, θ) represents the polar coordinates of the point Q. Since $r = \rho \sin \phi$ we obtain the following equations:

$$\begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \phi \end{cases}$$

The generic bounds for θ and ϕ are $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$, and the spherical coordinates of P are (ρ, θ, ϕ).

(ρ, θ, ϕ)

Ex1 The point $(4, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.



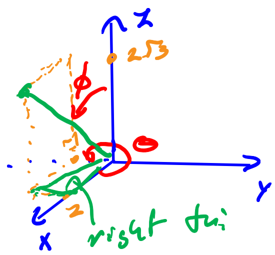
$z = \rho\cos(\phi) = 4\cos(\frac{\pi}{3}) = 2$

$y = \rho\sin(\phi)\sin(\theta) = 4\sin(\pi/3)\sin(\pi/4) =$

$x = \rho\sin(\phi)\sin(\theta) = 4\sin(\frac{\pi}{3})\cos(\frac{\pi}{4}) =$

In rectangular coords: (

Ex2 The point $(\sqrt{3}, -1, 2\sqrt{3})$ is given in rectangular coordinates. Find spherical coordinates for this point.



•) $\rho^2 = x^2+y^2+z^2 \Rightarrow \rho^2 = 3+1+12 \Rightarrow \boxed{\rho = 4}$

•) $z = \rho\cos\phi \Rightarrow \cos\phi = \frac{z}{\rho} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6}$

•) $x = \rho\sin(\phi)\cos(\theta) \Rightarrow \cos\theta = \frac{x}{\rho\sin\phi} = \frac{\sqrt{3}}{4(1/2)} = \frac{\sqrt{3}}{2} \Rightarrow \theta = 2\pi - \frac{\pi}{6}$

In spherical coords $(4, \frac{11\pi}{6}, \frac{\pi}{6})$

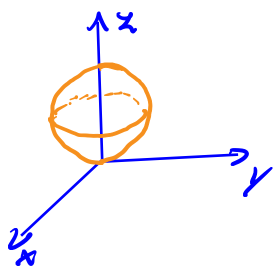
(ρ, θ, ϕ)

Ex3 Write in spherical coordinates the equations of the following surfaces:

a) Sphere: $\underbrace{x^2+y^2+z^2}_{\rho^2} = z.$

$\rho^2 = \rho\cos\phi$

$\rho = \cos\phi$



$x^2+y^2+(z-\frac{1}{2})^2 = 1/4$

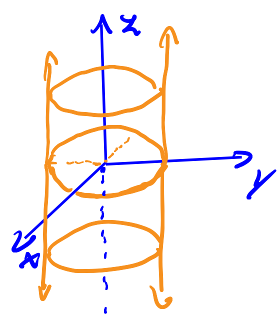
b) Cylinder: $x^2+y^2 = 4.$

$(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\cos\theta)^2 = 4$

$\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\cos^2\theta = 4$

$\rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta) = 4$

$\rho^2\sin^2\phi = 4 \Rightarrow \rho\sin\phi = 2$



$0 < \phi < \pi$

$\Rightarrow \sin\phi > 0$

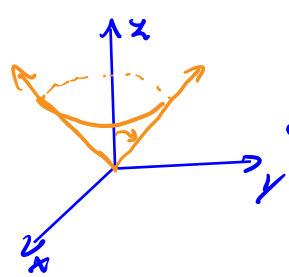
c) Cone: $z = \sqrt{x^2+y^2}.$

$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi}$

$\rho\cos\phi = \rho\sin\phi$

$1 = \frac{\sin\phi}{\cos\phi} = \tan\phi$

$\Rightarrow \phi = \frac{\pi}{4}$

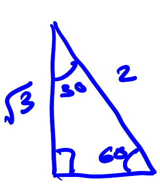


$0 < \phi < \pi/2$

$\sin\phi > 0$

$\cos\phi > 0$

d) TO DO. Cone: $z = \sqrt{3x^2+3y^2}$



$z = \sqrt{3}\sqrt{x^2+y^2} \Rightarrow \frac{1}{\sqrt{3}} = \tan\phi$

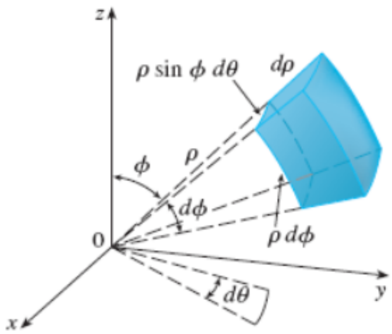
$\Rightarrow \phi = \frac{\pi}{6}$

Triple Integrals in Spherical Coordinates

Integration formula:

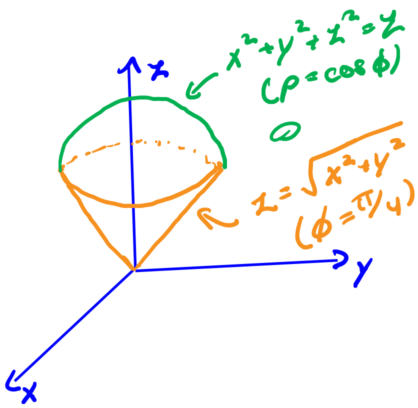
$$\iiint\limits_{E} f(x,y,z)\ dV = \iiint\limits_{E_{\rho,\phi,\theta}} \tilde{f}(\rho,\phi,\theta)\ \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$$

where $\tilde{f}(\rho,\phi,\theta) = f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$.

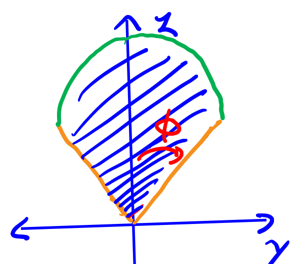


$$dV = \rho^2\ \sin\phi\ d\rho\ d\theta\ d\phi$$

Ex1. Set up a triple integral in spherical coordinates to compute the volume of the solid region bounded above by the sphere $x^2+y^2+z^2 = z$ and below by the cone $z = \sqrt{x^2+y^2}$. Also, evaluate the integral. Answer: $\pi/8$.

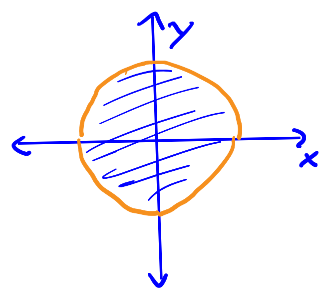


cross-section (yz-plane)



$\boxed{0 \le \phi \le \pi/4}$

projection onto xy-plane



$\boxed{0 \le \theta \le 2\pi}$

$$\rho \begin{cases} \text{enters: } \boxed{\rho = 0} \\ \text{leaves: } x^2+y^2+z^2 = z \Rightarrow \boxed{\rho = \cos\phi} \end{cases}$$

$$\text{Volume} = \iiint\limits_E 1\,dV = \int_0^{2\pi}\int_0^{\pi/4}\left(\int_0^{\cos\phi} 1\cdot\rho^2\sin\phi\,d\rho\right)d\phi\,d\theta$$

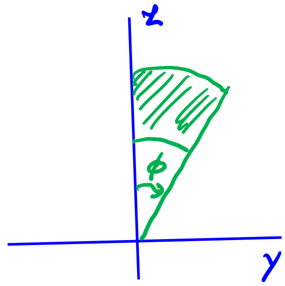
$$= \int_0^{2\pi}\int_0^{\pi/4}\left[\frac{\rho^3}{3}\sin\phi\right]_{\rho=0}^{\rho=\cos\phi} d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\frac{\cos^3\phi}{3}\sin\phi\,d\phi\,d\theta$$

$$= \ \ldots\ = \boxed{\frac{\pi}{8}}$$

Ex2. Let E be the solid region in the first octant that lies between the spheres $\rho = 2$ and $\rho = 4$ and above the cone $\phi = \pi/3$. Evaluate

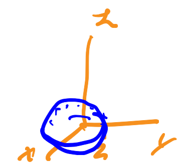
$$\iiint_E xyz\ dV$$



$\rho = 2$

$\rho^2 = 4$

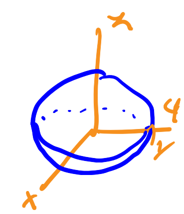
$x^2+y^2+z^2=4$

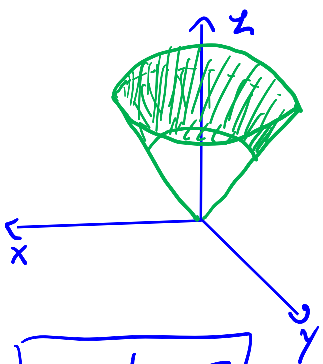


$\rho = 4$

$\rho^2 = 16$

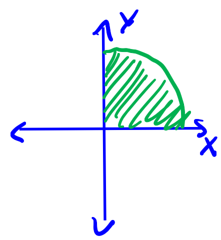
$x^2+y^2+z^2=16$





$$\boxed{0 \le \phi \le \frac{\pi}{3}}$$

Projection onto xy-plane



$$\boxed{0 \le \theta \le \frac{\pi}{2}}$$

$\rho \begin{cases} \text{enters: } \rho = 2 \\ \text{leaves: } \rho = 4 \end{cases}$

Now:

$$\iiint_E xyz\ dV = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{3}}\int_2^4 (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\ \rho^2\sin\phi\ d\rho\, d\phi\, d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{3}}\int_2^4 (\rho^5)(\sin^3\phi\cos\phi)(\cos\theta\sin\theta)\ d\rho\ d\phi\ d\theta.$$

$$= \left(\int_2^4 \rho^5 d\rho\right)\left(\int_0^{\pi/3} \sin^3\phi\cos\phi\ d\phi\right)\left(\int_0^{\pi/2}\cos\theta\sin\theta\ d\theta\right)$$

$$= \left(\frac{\rho^6}{6}\Big|_2^4\right)\left(\frac{\sin^4\phi}{4}\Big|_0^{\pi/3}\right)\left(\frac{\sin^2\theta}{2}\Big|_0^{\pi/2}\right)$$

$$= \dots = \frac{(2^6-1)(3)}{2^2} = \boxed{\frac{189}{4}}$$

Application of Triple Integrals
3D-Mass, Moments and Center of Mass

All the applications of double integrals can be immediately extended to triple integrals. For example, if the density function of a solid object that occupies the region E is $\delta(x, y, z)$ in units of mass per unit volume, at any given point (x, y, z), then its **mass** is

$$m = \iiint_E \delta(x, y, z)\ dV$$

and its **moment** about the three coordinate planes are

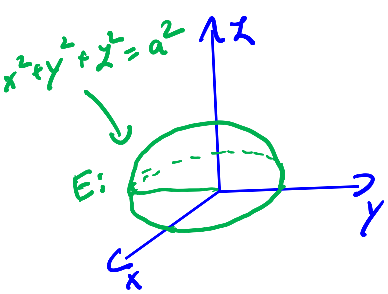
$$M_{yz} = \iiint_E x \cdot \delta(x, y, z)\ dV, \qquad M_{xz} = \iiint_E y \cdot \delta(x, y, z)\ dV$$

$$M_{xy} = \iiint_E z \cdot \delta(x, y, z)\ dV$$

The center of mass is located at the point $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{M_{yz}}{m}, \qquad \bar{y} = \frac{M_{xz}}{m}, \qquad \bar{z} = \frac{M_{xy}}{m}$$

Ex3. Find the center of mass of the upper solid hemisphere of radius a centered at the origin if the density at any point is proportional to ~~the~~ its distance from the base.



density : $\delta(x,y,z) = kz$ (k - constant)

we want $(\bar{x},\bar{y},\bar{z}) = \left(0, 0, \frac{M_{xy}}{\text{mass}}\right)$

•) $\text{mass} = \iiint_E (\text{density})\, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a k(\rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \frac{k\pi a^4}{4}$.

•) $M_{xy} = \iiint_E (z)(\text{density})\, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (\rho\cos\phi)(k\rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \frac{k(2\pi)a^5}{15}$

so $(\bar{x},\bar{y},\bar{z}) = \left(0, 0, \frac{k(2\pi)a^5/15}{k\pi a^4/4}\right) = \left(0, 0, \frac{8a}{15}\right)$.